Brick tilings – new insights

Shelomo I. Ben-Abraham

Department of Physics, Ben-Gurion University of the Negev,

IL-8410501 Beer Sheba, Israel

shelomo.benabraham@gmail.com

The three-dimensional *brick tiling and its color code* were introduced at ICQ 13 [1, 2] as a generalization of the *table tiling* and its color code [3]. Here we present their generalizations to arbitrary natural dimension. We also show how to reduce the color code.

In d dimensions, a *standard brick* (in what follows, simply *brick*) is a d-dimensional cuboid with edges of $2^0 = 1$, $2^1 = 2$, … , 2^{d-1} units. The *prototiles* of the brick tiling will be called *protobricks* and denoted by $\mathbf{B}_d$. A protobrick consists of $2^{d(d-1)/2}$ unit cubes $\mathbf{q}_d$. Thus, for instance, in dimensions 2, 3, 4 the number of unit cubes in a protobrick is 2, 8, 64, respectively. In d dimensions the protobricks come in $d!$ orientations labeled by *colors*. A g times inflated brick will be denoted $\mathbf{B}_d(g)$; thus, $\mathbf{B}_d(0) \equiv \mathbf{B}_d$ is a protobrick.

A maximal and faithful color code (alias labeling) of the unit cubes $\mathbf{q}_d$ in dD needs $d!2^{d(d-1)/2}$ "colors". That is $2\times2 = 4$ in 2D, $6\times8 = 48$ in 3D, $24\times64 = 1536$ in 4D, and so forth. Thus, while in principle possible, it is impracticable in any dimension higher than 3D.

While I can now in principle construct a brick tiling and its color code in any dimension I will focus on the brick tiling in 4D.

A brick tiling in any dimension dD can be constructed recursively. The inflated brick $\mathbf{B}_d(1)$ is partitioned along its longest edge L into its central half C and two peripheral quarters P, as shown in Fig. 1. A cut through C perpendicular to L reproduces the inflated brick $\mathbf{B}_{d-1}(1)$ of $(d-1)$D. A quarter P in dD contains 2^{d-2} parallel protobricks $\mathbf{B}_d$ with their longest edges perpendicular to L and their second longest edges parallel to L. This works in all dimensions including even 0D and 1D.

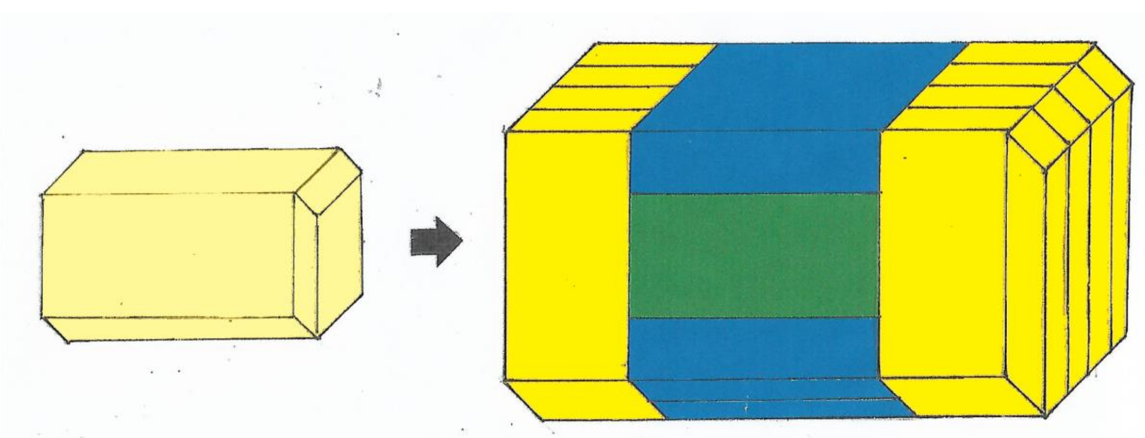

Figure 1. Two-dimensional isometric projection of four-dimensional brick inflation.

In dimensions 3D and higher the color code can be reduced to one half by making use of the mirror symmetries. However, this necessitates *seeding* an oriented and labeled protobrick. In 3D, the full code contains 48 labels (ik) ($i = 0, \ldots, 5$; $k = 0, \ldots, 7$). The reduction is performed by identifying $k = 6$ with 0, 7 with 1, 4 with 2 and 5 with 3. In dimensions higher than 3D it would be analogous, but as already said, it would be quite impractical.

1. S. I. Ben-Abraham & D. Flom, J. Phys.: Conf. Ser. **809** (2017) 012024.

2. D. Flom & S. I. Ben-Abraham, J. Phys.: Conf. Ser. **809** (2017) 012025.

3. E. A. Robinson Jr., Indag. Mathem. N.S. **10** (1999) 581-599.