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ELASTODYNAMIC GREEN'S FUNCTIONS FOR LAYERED STRUCTURES

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ABSTRACT

The Green's functions for a given medium are defined to be the solution of the wave propagation problem in presence of a concentrated force. Two analytical methods to derive the elastodynamic Green's functions using guided waves in layered structures are proposed. The dispersion relations and modal functions were obtained using the global matrix method and singular value decomposition (SVD) to find the null space vector. In the first approach, the representation theorem for elastodynamics was introduced to develop the modal expansion of the Green's function in terms of the analytical mode shapes normalized by a factor related to the power flow in the layered structure. In the second approach, temporal and spatial Fourier transforms were applied to the field variables and the boundary conditions. Application of the interface conditions and discontinuity conditions across the force lead to a system of linear equations for the calculation of the unknown constants. The residue theorem was applied for inverse Fourier transform of the matrix-form Green's function to recover the frequency domain expressions. In this paper, the displacements and stresses in a three-layered structure with different materials produced by a concentrated impulse load were obtained by the two methods. The problem was also solved using the conventional Finite Element Method (Abaqus). All three results were in a good agreement.

Keywords: Green's function, layered structures, global matrix method, representation theorem, lamb waves

1. INTRODUCTION

Wave propagation in layered media was first introduced by Thomson in 1950 [1] as the Transfer Matrix Method (TMM). This method relates the displacements and stresses on the top and bottom surfaces for all layers and applies continuity at the interface between layers to obtain the dispersion curves. An alternative method to the TMM was presented by Knopoff in 1964 [2]. Later, Mal [3] proposed the Global Matrix Method (GMM) to avoid the numerical instability at high frequencies as well as thicker layers presented in Knopoff's method. On the other hand, the extension of the representation theory to elastodynamics presented by Hudson and Knoppof [4] was used to obtain the Green's function for a multi-layered half space based on the modal functions in 1964 [5]. This paper provides a matrix form of the green's function based on two methods. In the first method, the residue theorem is applied to the global matrix. In the second method, the Green's function is obtained from the modal functions of the layered medium. The modal functions are calculated by Single Value Decomposition (SVD) in the global matrix. Finally, in order to validate both methods, the results are compared with conventional finite element method.

2. Green's Function Analytical Derivations

Two methods are provided to obtain the Green's function of layered structures. For the first method, the modal functions are obtained from the SVD of the global matrix. Then, the Green's function is obtained from representation theorem using the modal function. For the second method, the residue theorem is applied to the global matrix to transform the frequency wavenumber domain solution into frequency domain.

2.1 Representation Theorem

Consider a uniformly multilayered structure in which the elastic constants $\lambda(x_2)$, $\mu(x_2)$ and the density $\rho(x_2)$ are piecewise constant functions of x_2. Assume that the displacements and the stresses are only dependent of x_1 and x_2 .

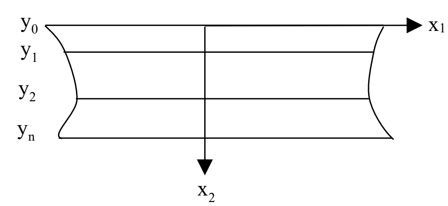

Figure 1: MULTILAYERD STRUCTURED

The displacement vector $\boldsymbol{u}(\boldsymbol{x})e^{-i\omega t}$ in each mode satisfies the equation of motion,

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$$\tau_{ij,j}(\boldsymbol{u}) + \rho\omega^2 u_i = 0 \qquad i,j = 1,2 \qquad (1)$$

at all points within the medium except at the interfaces. Furthermore, $u_i(\boldsymbol{x})$ and $\tau_{i2}(\boldsymbol{u})$ are continuous across each interface and $\tau_{i2}(\boldsymbol{u}) = 0$ on the top and bottom surfaces of the multilayered media. The wave displacements and stresses components can be expressed in the forms,

$$u_i^m(\boldsymbol{x}) = U_i^m(y)e^{ik_m x}$$
$$\tau_{ij}^m(\boldsymbol{x}) = T_{ij}^m(y)e^{ik_m x} \qquad (2)$$

where k_m, the wavenumber in the m[th] mode, is a root of the dispersion equation,

Let D denote the area enclosed by the straight lines C_0, C_1, C_-, C_+, figure (2). Since there are no body forces within D, the following identity can be easily proved.

$$\int_{C_0+C_-+C_1+C_+} \left(u_i^m \left(\tau_{ij}^n\right)^* - (u_i^n)^* \tau_{ij}^m\right)\upsilon_j ds = 0 \qquad (4)$$

in which υ_j is an outward unit normal and (*) indicates complex conjugate.

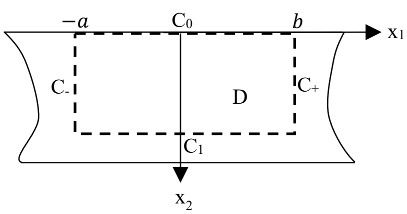

Figure 2: INTEGRATION CONTOUR

Let a time harmonic line force of unit magnitude at $x = \xi$ in a multilayered media. Let $G_{ij}(\boldsymbol{x}, \boldsymbol{\xi})$ denote the x_i component of the displacement produced at $\boldsymbol{x}$ due to a unit line force at $\boldsymbol{\xi}$ acting in the x_j direction. For fixed $\boldsymbol{\xi}$, $G_{ij}(\boldsymbol{x}, \boldsymbol{\xi})$ represents waves which propagate in the $+x_1$ direction if $x_1 > \xi_1$ and in the $-x_1$ direction for $x_1 < \xi_1$. Consequently, $G_{ij}(\boldsymbol{x}, \boldsymbol{\xi})$ may be expressed as,

$$G_{ij}(\boldsymbol{x}, \boldsymbol{\xi}) = \sum_m A_j(\xi_2) U_i^m(x_2) e^{ik_m(x_1-\xi_1)}, x_1 > \xi_1$$
$$G_{ij}(\boldsymbol{x}, \boldsymbol{\xi}) = \sum_m B_j(\xi_2) (U_i^m)^*(x_2) e^{-ik_m(x_1-\xi_1)}, x_1 < \xi_1 \qquad (5)$$

where $U_i^m(x_2)e^{ik_m x_1}$ is an eigenfunction of the multilayered media and A_j^m, B_j^m are, as yet, unknown functions of the source depth.

Recall the region D, which is bounded by the surfaces C_0, C_1, C_-, C_+. Let $U_i(\boldsymbol{x})$ denote a possible displacement field within D and on the bounding surfaces. According to the representation theorem of elastodynamics, the solutions inside D can be obtained in terms of the Green's function and the boundary data.

$$u_k(\boldsymbol{x}) = \int_{C_0+C_-+C_1+C_+} \left\{ G_{ik}(\boldsymbol{\xi}, \boldsymbol{x})\tau_{ij}(\boldsymbol{u}) - u_i(\boldsymbol{\xi})\tau_{ij}(\boldsymbol{G}_k)\right\} \nu_j ds(\xi) \qquad (6)$$

Since C_0 and C_1 are traction free,

$$u_k(\boldsymbol{x}) = \int_0^H \{ G_{ik}(\boldsymbol{\xi}, \boldsymbol{x})\tau_{i1}(\boldsymbol{u}) - u_i(\boldsymbol{\xi})\tau_{i1}(\boldsymbol{G}_k)\}_{\xi_1=-a} ds(\xi_2) + \int_0^H \{ G_{ik}(\boldsymbol{\xi}, \boldsymbol{x})\tau_{i1}(\boldsymbol{u}) - u_i(\boldsymbol{\xi})\tau_{i1}(\boldsymbol{G}_k)\}_{\xi_1=b} ds(\xi_2) \qquad (8)$$

Let $u_k(\boldsymbol{x}) = U_k^n(x_2)e^{ik_n x_1}$ be a wave eigenfunction for multilayered media propagating in the positive x_1-direction. Note that on $x_1 = -a$, $x_1 < \xi_1$, so that

$$G_{ik}(\boldsymbol{\xi}, \boldsymbol{x}) = \sum_m A_k^m(x_2) U_i^m(\xi_2) e^{ik_m(\xi_1-x_1)} \qquad (9)$$

and on $x_1 = b$, $x_1 > \xi_1$, so that

$$G_{ik}(\boldsymbol{\xi}, \boldsymbol{x}) = \sum_m B_k^m(\xi_2) (U_i^m)^*(x_2) e^{-ik_m(\xi_1-x_1)} \qquad (10)$$

Substituting from (9) and (10) into (8) we have,

$$B_k^n(\xi_2) = \frac{1}{R^n} U_k^n(x_2) \qquad (11)$$

Similarly, by assuming that $u_k(\boldsymbol{x}) = (U_k^n(x_2))^* e^{-ik_n x_1}$, a wave eigenfunction propagating in the negative x_1-direction, in (8), it can be proved that,

$$A_k^n(\xi_2) = \frac{1}{R^n} (U_k^n(x_2))^* \qquad (12)$$

Using (11) and (12) in (5), the wave terms of the Green's function may be written in the form,

$$G_{ij}(\boldsymbol{x}, \boldsymbol{\xi}) = \sum_m \frac{1}{R^m} U_i^m(x_2) (U_j^m)^*(\xi_2) e^{ik_m(x_1-\xi_1)}, x_1 > \xi_1$$
$$G_{ij}(\boldsymbol{x}, \boldsymbol{\xi}) = \sum_m \frac{1}{R^m} (U_i^m)^*(x_2) U_j^m(\xi_2) e^{-ik_m(x_1-\xi_1)}, x_1 < \xi_1 \qquad (13)$$

Applying the SVD to the global matrix, one can obtained the approximate modal function to calculate the Green's function for a multilayered structure.

2.2 Residue Calculations

Defining layered matrix:

$$Q(m) = \begin{bmatrix} ik & \eta_2 & ik & -\eta_2 \\ -\eta_1 & ik & \eta_1 & ik \\ -2ik\eta_1\mu & -\zeta_2\mu & 2ik\eta_1\mu & -\zeta_2\mu \\ \zeta_2\mu & -2ik\eta_2\mu & \zeta_2\mu & 2ik\eta_2\mu \end{bmatrix} \qquad (1)$$

$$E(m) = diag(e^{-\eta_1 h_m} \quad e^{-\eta_2 h_m}) \tag{2}$$

$$C(m) = \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} \tag{3}$$

Therefore, the modal function for each layer m can be express as:

$$U(m) = \begin{Bmatrix} U \\ V \\ T \\ \Sigma \end{Bmatrix} = Q(m)E(x_2, m)C(m)e^{ikx_1} \tag{4}$$

By applying proper boundary condition, the global matrix can be assembled by:

$$G = \begin{Bmatrix} Q_{21}(1) & Q_{22}(1)E(1) & Z & Z & \dots & Z & Z \\ Q_{11}(1)E(1) & Q_{12}(1) & -Q_{11}(2) & -Q_{12}(2)E(2) & \dots & Z & Z \\ Q_{21}(1)E(1) & Q_{22}(1) & -Q_{21}(2) & -Q_{22}(2)E(2) & \dots & Z & Z \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Z & Z & Z & Z & \dots & Q_{21}(N)E(N) & Q_{22}(N) \end{Bmatrix} \tag{5}$$

In the frequency wavenumber domain, the boundary condition can be expressed as:

$$C = G^{-1}F \tag{6}$$

One can obtained the C vector in frequency domain by applying residue theorem:

$$C = \frac{1}{2\pi}(2\pi i)\left(\sum \frac{MF}{\frac{\partial \det(G)}{\partial k}}\right) \tag{7}$$

Finally, the matrix form of Green's function can be obtained by letting F be a unit impulse function and splitting the following matrix function:

$$U(m) = \begin{Bmatrix} U \\ V \\ T \\ \Sigma \end{Bmatrix} = iQ(m)E(x_2, m)\frac{MF}{tr(M\dot{G})}e^{ikx_1} \tag{8}$$

3. RESULTS AND DISCUSSION

Results from the two proposed methods are compared on a single layer aluminum plate subjected to a buried impulse load.

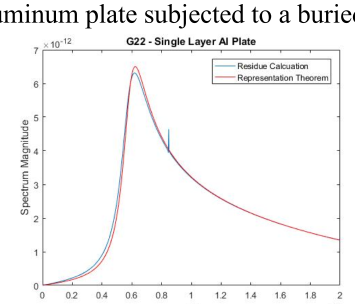

Figure 3: GREEN'S FUNCTION SPECTRUM ON TOP SURFACE FOR A BURIED LOAD – S0 MODE

Results from the Global Matrix Method is obtained for a three-layer non-homogeneous structure subjected to an impulse load on the free surface.

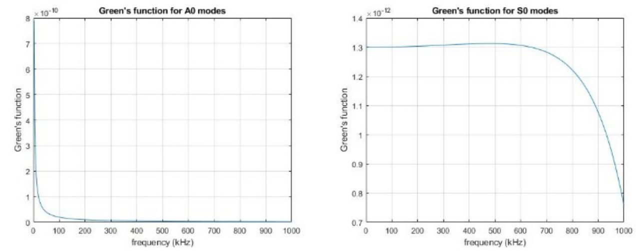

Figure 4: GREEN'S FUNCTION SPECTRUM FOR TWO FUNDAMENTAL MODES

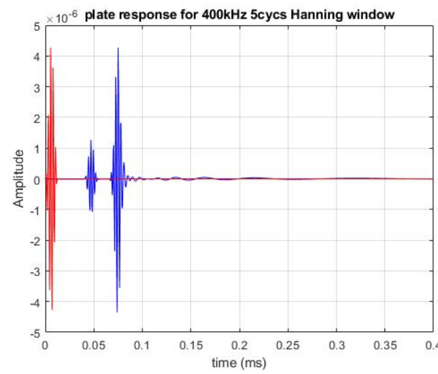

Figure 5: PLATE RESPONSE FOR 400kHz 5CYCS HANG-WINDOW EXCITATION SIGNAL

4. CONCLUSION

The results obtained from the two analytical approaches and those from the numerical simulations show excellent agreement.

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