SHEAR MODULI UNDER UNIAXIAL LOADING : Stress Dependence and Isotropy Condition Relating Shear Moduli to Young's Modulus and Poisson's Ratio

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ABSTRACT

This paper presents stress/strain dependence of shear moduli under uniaxial homogeneous loading in direction 3 of an elastic solid, which has initially an isotropic or cubic symmetry. The stress/strain dependence of the Young's modulus and Poisson's ratio under uniaxial loading was previously treated in detail by this author and Sachse [1] for a similar case. Stress/strain dependence of shear moduli is expressed in terms of thermodynamic elastic coefficients and uniaxial stress $\sigma_3 = \sigma_{33}$. The thermodynamic elastic coefficients are then expressed in terms of the second-order elastic constants (SOEC) and the third-order elastic constants (TOEC). The relation between the Young's modulus $E(\mathbf{a})$, Poisson's ratio $\nu(\mathbf{a})$, and shear modulus $G(\mathbf{a})$ at stress-free isotropic state is given by $G(\mathbf{a}) = E(\mathbf{a})/2[1+\nu(\mathbf{a})]$, where vector **a** represents a stress-free natural state. This relation no longer holds valid for a stressed media. However, an isotropic solid at stress-free state becomes transversely isotropic under uniaxial loading around the loading direction 3 and a similar relation holds at a uniaxially stressed state **X**. Letting shear modulus $G_{66}(\mathbf{X})$ denote a shear modulus defined by infinitesimal Cauchy shear stress $d\sigma_6(\mathbf{X})$ divided by infinitesimal shear strain $d\varepsilon_6(\mathbf{X})$, a relation $G_{66}(\mathbf{X}) = E_3(\mathbf{X})/2[1+\nu(\mathbf{X})]$ holds at a stressed state **X.** A similar relation fails to hold for other shear moduli $G_{44}(\mathbf{X})$ and $G_{55}(\mathbf{X})$ defined similar to $G_{66}(\mathbf{X})$.

Keywords: thermodynamic elastic stiffness coefficients, effective elastic stiffness and compliance coefficients, effective shear modulus, effective Young's modulus, effective Poisson's ratio.

NOMENCLATURE

i,j,k,l	When used as a subscript, they denotes Cartesian coordinate indices equal to 1,2, or 3.
μ, ν	When used as a subscript, they denote Voigt indices equal to 1,2,⋯, or 6.
a	Coordinates of stress-free natural state. Vector **a** also represents the stress-free natural state.
X	Coordinates of stressed initial state. Vector **X** also represents a stressed initial state.
σ_{ij} & σ_μ	Cauchy stress
ε_{ij} & ε_μ	Infinitesimal strain
η_{ij} & η_μ	Lagrangian strain
C_{ijkl} & $C_{\mu\nu}$	Thermodynamic elastic stiffness coefficient
K_{ijkl} & $K_{\mu\nu}$	Effective elastic stiffness coefficient
Q_{ijkl} & $Q_{\mu\nu}$	Effective elastic compliance coefficients
G_{ijkl} & $G_{\mu\nu}$	Effective shear modulus
E_3	Effective Young's modulus in the loading direction 3.
ν	Effective Poisson's ratio

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1. INTRODUCTION

Stress or strain dependence of the Young's modulus $E_3(\mathbf{X})$ and Poisson's ratio $\nu(\mathbf{X})$ of an elastic solid, which has isotropic or cubic symmetry at stress-free natural state and is subsequently uniaxially loaded in X_3 direction, was investigated in detail by this author and Sachse in Ref. [1] using the finite deformation theory. Loading direction 3 is chosen to be parallel to one of cubic axes, say [001] in the case of a cubic crystal. In a Cartesian coordinate system that includes loading direction X_3, $\sigma_{ij} = \sigma_{33}\delta_{i3}\delta_{j3}$ and $\sigma_\mu = \sigma_3\delta_{\mu 3}$, $\varepsilon_{ij} = \varepsilon_{ij}\delta_{ij}$ and $\varepsilon_4 = \varepsilon_5 = \varepsilon_6 = 0$, where δ_{ij} and $\delta_{\mu 3}$ denote Kronecker's delta. In other words, no shear stresses and shear strains are generated in the uniaxial loading and stress/strain dependence of shear moduli G_{ijkl} $(\mathbf{X})$ & $G_{\mu\nu}$ $(\mathbf{X})$ are left unexplored. However, these material properties of shear moduli vary under stress. In this work we investigate the stress dependence of G_{ijkl} $(\mathbf{X})$ and $G_{\mu\nu}$ $(\mathbf{X})$ under uniaxial loading. They are expressed in terms of the second-order and third-order elastic constants of a material. In static mechanics it is well-known that the Young;s modulus, Poisson's ration and shear modulus of an isotropic solid at a stress-free state are related by the formula $G(\mathbf{a}) = E(\mathbf{a})/2[1+ \nu(\mathbf{a})]$. This relation breaks down under general stresses. However, under uniaxial stress applied to an isotropic solid in direction X_3, the X_1X_2 plane maintains transverse isotropy around X_3 axis and a similar relation $G_{66}(\mathbf{X}) = E_3(\mathbf{X})/2[1+ \nu(\mathbf{X})]$ holds valid, where G_{66} $(\mathbf{X}) = d\sigma_6/d\varepsilon_6$. However, a similar relation does not apply to other shear moduli, $G_{44}(\mathbf{X}) = G_{55}(\mathbf{X}) \neq G_{66}$.

2. THEORETICAL METHOD

One of the most useful quantities in describing the behaviors of elastic moduli of solids under stress is the effective elastic stiffness coefficients K_{ijkl} & $K_{\mu\nu}$ and the effective elastic compliance coefficients Q_{ijkl} and $Q_{\mu\nu}$, which are expressed by Thurston [2] and Wallace [3] as

$$K_{ijkl}(\mathbf{X}) = \left(\partial\sigma_{ij}/\partial\varepsilon_{kl}\right)_{\mathbf{X}} = C_{ijkl}(\mathbf{X}) - \sigma_{ij}(\mathbf{X})\delta_{kl} + (1/2)\left[\sigma_{ik}(\mathbf{X})\delta_{jl} + \sigma_{il}(\mathbf{X})\delta_{jk}\right] + (1/2)\left[\sigma_{jk}(\mathbf{X})\delta_{il} + \sigma_{jl}(\mathbf{X})\delta_{ik}\right] \quad (1)$$

$$Q_{ijkl}(\mathbf{X}) = \left(\partial\varepsilon_{ij}/\partial\sigma_{kl}\right)_{\mathbf{X}} \quad (2)$$

$K_{\mu\nu}$ is given in a matrix form for an orthotropic medium [4] and for a generally anisotropic medium [5]. For a medium with cubic or higher symmetry at stress-free natural state, which is loaded with the uniaxial stress $\sigma_\mu = \sigma_3\delta_{\mu 3}$ in X_3 direction, $K_{\mu\nu}(\mathbf{X})$ in terms of the thermodynamic elastic stiffness coefficients $C_{\mu\nu}(\mathbf{X})$and σ_3 can be obtained from Eq. (86) of Ref. 4 by putting $\sigma_1 = \sigma_2 = 0$ and $C_{11} = C_{22}$, $C_{44} = C_{55}$, and $C_{13} = C_{23}$. Three diagonal elements of $K_{44} = K_{55} = C_{44} + \sigma_3/2$ and $K_{66} = C_{66}$ represent two distinctive shear moduli under uniaxial loading, $G_{44} = G_{55} = C_{44} + \sigma_3/2$ and $G_{66} = C_{66}$. For an isotropic solid at stress-free natural state, identities $C_{12} = C_{13}$, $C_{44} = C_{66}$, and $2C_{44} = C_{11} - C_{12}$ hold. Therefore there are only two independent elastic constants for an isotropic solid and last identity $2C_{44} = C_{11} - C_{12}$ is usually called "isotropy condition". Using the effective elastic compliance coefficients at stress-free natural state the isotropic identity is translated into

$$2\left[Q_{11}(\mathbf{a}) - Q_{12}(\mathbf{a})\right]/Q_{44}(\mathbf{a}) = 2Q_{11}(\mathbf{a})\left[1 - Q_{11}^{-1}(\mathbf{a})\,Q_{12}(\mathbf{a})\right]/Q_{44}(\mathbf{a}) = 1 \quad (3)$$

In Eq. 3, we note that $Q_{11}(\mathbf{a}) = 1/E(\mathbf{a})$, the reciprocal of the Young's modulus, $Q_{11}^{-1}(\mathbf{a})\,Q_{12}(\mathbf{a}) = \nu(\mathbf{a})$, Poisson's ratio and $Q_{44}(\mathbf{a}) = 1/G(\mathbf{a})$, the reciprocal of the shear modulus. Therefore the isotropy condition at the stress-free natural state is expressed by the following relation

$$G(\mathbf{a}) = E(\mathbf{a})/2(1 + \nu), \quad (4)$$

which is a well-known formula in static mechanics. When an isotropic solid is uniaxially stressed, general isotropic condition breaks down. However, it still maintains transverse isotropy about the loading direction. This transverse isotropy requires $2K_{66}(\mathbf{X}) = K_{11}(\mathbf{X}) - K_{12}(\mathbf{X})$, which is expressed via Eq. 1 as $2C_{66} = C_{11} - C_{12}$. Proceeding in a similar way yields the following transverse isotropy relation

$$G_{66}(\mathbf{X}) = E_3(\mathbf{X})/2[1 + \nu(\mathbf{X})] \quad (5)$$

However, for other shear moduli $G_{44}(\mathbf{X}) = G_{55}(\mathbf{X}) \neq E_3(\mathbf{X}/2[1 + \nu(\mathbf{X})]$.

2.1 Stress Dependence of Shear Moduli

Variations of shear moduli $G_{66}(\mathbf{X}) = C_{66}(\mathbf{X})$ and $G_{44}(\mathbf{X}) = G_{55}(\mathbf{X}) = C_{44}(\mathbf{X}) + \sigma_3/2$ with the applied uniaxial stress σ_3 can be obtained by expressing the thermodynamic stiffness coefficients $C_{44}(\mathbf{X})$ and $C_{66}(\mathbf{X})$ in terms of the second-order and third-order elastic constants. Referring to Eq. (19.15) of Ref. 5 and denoting the principal stretches along the X_1 direction and loading direction X_3 by λ_1 $(= \lambda_2)$ and λ_3, respectively, one obtains for a cubic solid at a stressed state $\mathbf{X}$

$$G_{66}(\mathbf{X}) = C_{66}(\mathbf{X}) = C_{1212}(\mathbf{X}) = \lambda_1^2\lambda_1^{-1}[C_{44}(\mathbf{a}) + C_{155}\eta_{11} + C_{144}\eta_{33} + \cdots] \quad (6)$$

$$G_{44}(\mathbf{X}) = G_{55}(\mathbf{X}) = C_{2323}(\mathbf{X}) + \sigma_3/2 = \lambda_3[C_{44}(\mathbf{a}) + (C_{144} + C_{155})\,\eta_{11} + C_{155}\eta_{33} + \cdots] + \sigma_3/2 \quad (7)$$

Making use of relations $\lambda_1^2 = 1+2\,\eta_{11}$, $\lambda_3 = (1 + 2\eta_{33})^{1/2}$, $\eta_{11} \cong S_{12}\sigma_3 + \cdots$, $\eta_{33} \cong S_{11}\sigma_3 + \cdots$, where S_{11} and S_{12} are elastic compliance constants at a stress-free state **a**, Eqs. 6 and 7 yield the shear moduli to the first order of σ_3 as a function of the applied uniaxial stress σ_3 as below:

$$G_{66}(\mathbf{X}) = C_{44}(\mathbf{X}) + [C_{44}(\mathbf{a})(2S_{12} - S_{11}) + S_{11}C_{144} + 2S_{12}C_{155} + \cdots]\,\sigma_3 \quad (8)$$

$$G_{44}(\mathbf{X}) = G_{55}(\mathbf{X}) = C_{44}(\mathbf{a}) + [(1/2) + S_{11}C_{44}(\mathbf{a}) + (S_{11}+S_{12})\,C_{155} + S_{12}\,C_{144} + \cdots]\sigma_3 \quad (9)$$

In the above Eqs 6-9, the third-order elastic constants are defined at a stress-free natural state. For an isotropic solid at a stress-free state $C_{144} = C_{155} = (1/2)\,(C_{112} - C_{123})$.

3. RESULTS AND DISCUSSION

Nonlinear elastic behavior of the shear moduli are illustrated with examples of cubic silicon and isotropic polystyrene polymer. Hall [6] reported the elastic constants data of cubic silicon in units of GPa : $C_{11}(\mathbf{a}) = 165.64$, $C_{12}(\mathbf{a}) = 63.94$, $C_{44}(\mathbf{a}) = 79.51$, $C_{111}(\mathbf{a}) = -795$, $C_{112}(\mathbf{a}) = -445$, $C_{123}(\mathbf{a}) = -75$, $C_{144}(\mathbf{a}) = 15$, $C_{155}(\mathbf{a}) = -310$, and $C_{456}(\mathbf{a}) = -86$. From these data one obtains in units of $(\mathrm{GPa})^{-1}$ $S_{11} = 7.690549 \times 10^{-3}$ and $S_{12} = -2.137349 \times 10^{-3}$. Under an applied stress σ_3= 1 GPa , $G_{66}(\mathbf{X}) = 79.9988$ GPa, which is a small increase of 0.615 %, and $G_{44}(\mathbf{X}) = G_{55}(\mathbf{X}) = 78.86678$ GPa, which is 0.809 % decrease from the stress-free state value 79.51 GPa of $G_{44}(\mathbf{a}) = G_{55}(\mathbf{a})$. For a relatively soft isotropic polystyrene polymer, its elastic constants are reported by Hughes and Kelly [7] to be in units of GPa: $C_{11}(\mathbf{a}) = 5.651$, $C_{12}(\mathbf{a}) = 2.889$, $C_{44}(\mathbf{a}) = 1.381$, $C_{111}(\mathbf{a}) = -91$, $C_{112}(\mathbf{a}) = -37.80$, and $C_{123}(\mathbf{a}) = -21.20$. . From these data one obtains in units of $(\mathrm{GPa})^{-1}$ $S_{11} = 0.27054$ and $S_{12} = -0.091520$. Under a low applied stress σ_3= 0.1 GPa = 100 MPa, Eqs. 8 and 9 yields $G_{66}(\mathbf{X}) = 1.20217$ GPa, which is a relatively large decrease of 12.95 % from the shear modulus at a stress-free state and $G_{44}(\mathbf{X}) = G_{55}(\mathbf{X}) = 1.352165$ GPa, which is a small decrease of 2.09 % from the shear modulus at a stress-free state.

4. CONCLUSION

This work shows the stress dependence of the shear moduli of cubic and an isotropic solids at a stress-free natural state, which was subsequently uniaxially loaded in the direction of one of the cubic axes or in any direction of isotropic material at a stress-free state. Isotropy condition at a stress-free state and transverse isotropy condition under uniaxial loading yield a relation between the shear modulus, Poisson's ratio and Young's modulus.

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