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A MODIFIED NORMAL-MODE EXPANSION METHOD FOR FORCED-GUIDED WAVES IN PLATES ---- SHEARED HORIZONTAL WAVE

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ABSTRACT

Guided waves in an elastic plate are called forced-guided waves if they are generated by body forces and/or surface tractions. The normal-mode expansion method has been used extensively by numerous researchers and practitioners to solve for such forced-guided wave problems in plates. Using horizontally polarized shear waves as examples, this paper demonstrates that solutions from the normal-mode expansion method often do not describe fully the actual elastodynamic field in the plate. To improve upon it, this paper further develops a modified normal-mode expansion method that yields solutions that satisfy all elastodynamic equations including the Hooke's law and prescribed traction boundary conditions, thus providing a full description of the elastodynamic field in the plate.

Keywords: modified normal mode expansion, forced guided wave, sheared horizontal wave

NOMENCLATURE

ρ	density
μ	elastic constant
u	displacement
σ	stress
k	wavenumber
ω	circular frequency
i	unit of imaginary number

1. INTRODUCTION

Waveguide is a common device used in many engineering fields including optics, acoustics, microelectronics, etc. Guided waves can be generated in a waveguide by a distributed body force in and/or by tractions prescribed on the surfaces of the waveguide. Such guided waves are called the forced-guided waves. To solve for the forced-guided wave problems, Auld in his classic monograph [1] developed the normal-mode expansion (NME) method, which has been proven to be a powerful tool for solving forced guided wave problems in a plate. The method was rederived rigorously by Kino in his 1987 book [2] with a focus on elastic waves.

However, as well be shown in this paper, solutions from the classical NME method do not necessarily satisfy (i) the Hooke's law and (ii) the prescribed traction boundary conditions. Therefore, they are not the exact elastodynamic solutions of the corresponding forced-guided wave problems. In fact, the classical NME solution could be very far off from the exact solution, by more than 50% in certain cases. To improve upon the classical NME method, we will develop in this paper a modified NME method that yields exact elastodynamics solutions in that they satisfy all governing equations and prescribed boundary conditions in elastodynamics. This paper will focus on the case of transversely polarized shear (SH) waves. The Rayleigh-Lamb waves will be discussed in a separate publication.

2. PROBLEM STATEMENT

An elastic plate with thickness $2h$ is considered in this problem. The xoy plane in a Cartesian coordinate system is used and $y = \pm h$ are the surfaces of the plate. The plate is characterized by mass density ρ and elastic constant μ.

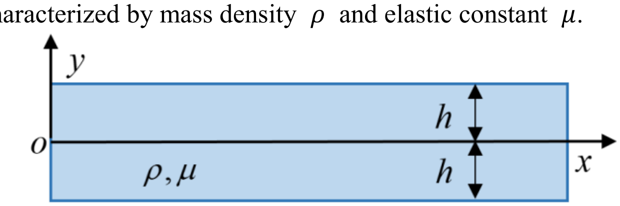

FIGURE 1: SKETCH OF A PLATE WITH THICHNESS $2h$.

In the absence of body forces and surface traction, the free SH Modes in the plates can be written as [3],

$$u = u_m(y)e^{i(k_m x - \omega t)} \quad , \tag{1}$$

where u is the displacement in the transverse direction, and

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$$u_m(y) = A_s \cos(\xi_m y),\ \ u_m(y) = A_a \sin(\xi_m y) \quad (2)$$

are the symmetric and anti-symmetric modes, respectively. In the above,

$$\xi_m = \sqrt{\left(\frac{\omega}{c_T}\right)^2 - k_m^2}\ . \quad (3)$$

where, for convenience, we chose

$$A_s = A_a = \frac{2h}{\pi}. \quad (4)$$

The shear stresses corresponding to (1) can be written as $\boldsymbol{\tau} = \boldsymbol{\tau}_m(y)e^{ik_m x}$, where

$$\boldsymbol{\tau}_m(y)\cdot\hat{\mathbf{x}} = ik_m \mu u_m(y)\ ,\ \ \boldsymbol{\tau}_m(y)\cdot\hat{\mathbf{y}} = \mu\frac{du_m(y)}{dy}\ . \quad (5)$$

with $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ being the unit normal vectors in the *x*- and *y*-directions. Traction-free boundary conditions dictate that,

$$\xi_m h = (m-1)\pi\ ,\ \ \xi_m h = \left(m - \frac{1}{2}\right)\pi\ ,\ \ m = 1,2,3\cdots \quad (6)$$

for the symmetric and anti-symmetric modes, respectively.

3. MODIFIED NORMAL MODE EXPANSION METHOD

To solve the forced-guided wave problem, it is convenient to decompose the body force and the surface traction into symmetric and anti-symmetric cases, which correspond to the symmetric and anti-symmetric modes, respectively. Only the symmetric case will be considered here. For brevity, we will consider only a special case where the plate is subjected to a non-zero body force but zero-traction on the upper and lower surfaces, i.e.,

$$f = f^s(x,y) = \frac{\pi\mu}{2h}e^{ik_r x}\ ,\ \ p_\pm(x) = p_\pm^s(x) = 0\ \ , \quad (7)$$

where, for the purpose of this example, we assume that (ω, k_r) is not on any of the dispersion curves. For this problem, it can be easily verified that the exact solution is given by

$$u = \frac{\pi}{2h}\frac{e^{ik_r x}}{k_r^2 - k_1^2}\ \ , \quad (8)$$

where $k_1 = \omega/c_T$ is the wavenumber of the first propagating symmetric mode. In other words, Eq. (8) satisfies the equation of motion, the Hooke's law and the zero-traction boundary conditions.

In the proposed modified NME method, we assume that

$$u = \sum_{n=1}^{\infty} a_n(x) u_m(y) \quad (9)$$

By applying the Hooke's law, we obtain the corresponding shear stresses

$$\boldsymbol{\tau}\cdot\hat{\mathbf{x}} = \sum_m a_m(x)\boldsymbol{\tau}_m(y)\cdot\hat{\mathbf{x}} + \mu\sum_m A_m(x)u_m(y)\ , \quad (10)$$

$$\boldsymbol{\tau}\cdot\hat{\mathbf{y}} = \sum_m a_m(x)\boldsymbol{\tau}_m(y)\cdot\hat{\mathbf{y}}\ . \quad (11)$$

where

$$A_m(x) = \frac{da_m(x)}{dx} - ik_m a_m(x)\ . \quad (12)$$

We note that (10)-(11) differ from their counterparts in the classical NME method due to the additional term related to $A_m(x)$.

Making use of (9)-(12) in the reciprocity equation [1], we arrive at a differential equation for $a_n(x)$,

$$4P_{nN}\frac{\partial}{\partial x}\{a_n(x)e^{-ik_N^* x}\} - i\omega Q_{nN}\frac{\partial}{\partial x}\{e^{i(k_n - k_N^*)x}\frac{\partial}{\partial x}[a_n(x)e^{-ik_n x}]\}$$
$$= i\omega g_N(x)e^{-ik_N^* x} \quad (13)$$

where $n = N = 1,2,\cdots$, and

$$P_{11} = A_s^2\mu\omega h k_1\ ,\ \ Q_{11} = 2A_s^2\mu h\ ,$$
$$P_{nN} = \frac{A_s^2\mu\omega h}{4}(k_n + k_N^*)\ \ ,\ \ Q_{nN} = A_s^2\mu h\ \ ,$$
$$g_1 = A_s\pi\mu e^{ik_r x}\ \ ,\ \ g_{N+1} = 0\ \ ,\ \text{for}\ \ n > 1 \quad (14)$$

In the above, $k_n = k_N$ for propagating modes and $k_n = -k_N$ for evanescent modes.
The solution of (13) gives,

$$a_1(x) = \frac{\pi}{2A_s h(k_r^2 - k_1^2)}e^{ik_r x}\ \ , a_n(x) = 0\ ,\ \ n > 1 \quad (15)$$

Finally, the displacement follows from (9)

$$u = a_1(x)u_1(y) = \frac{\pi}{2h}\frac{e^{ik_r x}}{k_r^2 - k_1^2}\ . \quad (16)$$

This is identical to the exact solution given in (8). In other words, the modified NME method gives the exact solution to this problem, as expected.

Next, let us use the classical NME method to solve the same problem. It can be shown that the classical NME method yields

$$u = a_1(x)u_1(y) = \frac{\pi}{4h}\frac{e^{ik_r x}}{(k_r - k_1)k_1}\ . \quad (17)$$

Comparing (17) with the exact solution (8) gives the relative error in the classical NME solution,

$$\left|\frac{u(\text{exact}) - u(\text{classical NME})}{u(\text{exact})}\right| = \frac{1}{2}\left|\frac{k_r}{k_1} - 1\right|. \quad (18)$$

Clearly, the error could be significant depending on how different k_r is from k_1. For example, if $k_r = 2k_1$, the error will be $\Delta = 50\%$.

4. CONCLUSION

It is shown in this paper that the solutions from the classical NME method often do not satisfy the Hooke's law and the prescribed traction boundary conditions, therefore, do not yield the exact elastodynamic solution for the forced-guided wave problems. In certain cases, the classical NME solution could be significantly different from the exact solution. To improve upon the classical NME method, this paper developed a modified NME method that yields the exact solution for forced-guided wave problems.

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